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Radial basis function networks for delay differential equation

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Abstract In this article, a numerical technique is developed for solving delay differential equations. The proposed method combines the method of steps with the radial basis function networks. A delay differential equation is transformed to an ordinary differential equation and then the radial basis function collocation method is implemented to find the solution of the ordinary differential equation. The procedure for implementation of the radial basis function collocation method and the proposed method for delay differential equation is presented in detail. Numerical experiments are carried out on a number of examples to show the advantages of the proposed technique over the radial basis function collocation method.

Mathematics Subject Classification 65L60 · 65N35

المخلص

في هذا البحث، نقترح تقنية لحل معادلات تأخير تفاضلية. التقنية المقترحة مركبة من طريقة من خطوات وشبكات دالة أساس إشعاعي. وفقًا لهذه التقنية، نحول معادلة التأخير التفاضلية إلى معادلة تفاضلية عادية ثم نطبق طريقة انتظام دالة الأساس الإشعاعي لإيجاد حل للمعادلة التفاضلية العادية التي تم الحصول عليها. تم تقديم عملية تطبيق طريقة انتظام دالة الأساس الإشعاعي والطريقة المقترحة لمعادلة التأخير التفاضلية. تم إعطاء أمثلة عددية لإثبات بعض المميزات للتقنية المقترحة على طريقة انتظام دوال الأساس الإشعاعي.

1 Introduction

A delay differential equation is a generalization of the ordinary differential equation, which is suitable for a physical system that depends on the past history. The solution of delay differential equations not only requires the information of the current state, but also requires some information about the previous state. Delay differential equations have numerous applications in mathematical modelling [1]. During the last decade, several researchers have devoted their expertise to the development of numerical techniques for delay differential equations. As a result, different numerical methods have been developed and applied for providing approximate solutions. For details, we refer the reader to [2–4] and references therein.

The method of steps, described in [5], is the simplest method used to convert a delay differential equation to a non-delay differential equation. In [6], the method of steps is utilized for solving integer order delay differential equations.

Over the past decade, major advances have occurred in applications of radial basis function collocation method for solving differential equations arising from mathematical problems. The radial basis function is

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a mesh-free scheme, which avoids grid generation and the domain of interest can be considered by a set of scattered data points [7]. The multiquadric radial basis functions [8,9] is very important and useful method for the numerical solution of ordinary and partial differential equations. On the basis of numerical and theoretical evidences, it has been shown in [10,11] that the radial basis function collocation method is very accurate even for a small number of collocation points.

In the present work, we have combined the method of steps with the radial basis function collocation method to develop a technique for solving delay differential equations. The total number of collocation points and radial basis functions are taken to be same. Comparison of solutions by the proposed method with the radial basis function collocation method and with exact solution is presented.

2 Radial basis function networks

We can expand any function $f(x) \in L^2[a, b]$ into radial basis function networks as,

$$f(x) \approx \sum_{j=0}^M w_j h_j(x), \quad (2.1)$$

where M is the number of radial basis functions, w_j is the network weight, and x is the input. In the present work, we consider a multiquadrics radial basis function

$$h_j(x) = \sqrt{(x - d_j)^2 + b_j^2}$$

where d_j s are the centres and b_j s are the widths of radial basis function. Large or small width of the radial basis function makes the response of a neuron flat or peak, respectively. The width of the j th radial basis function is chosen as in [8], that is

$$b_j = \nu e_j,$$

where $\nu > 0$ and e_j is the distance from the j th centre to the nearest centre. For input points $\{x_j\}_{j=0}^M$, where M is the total number of collocation points, we get desired output $\{y_j\}_{j=0}^M$ corresponding to the collocation points. The set of network weights $\{w_j\}_{j=0}^M$ can be found using the general linear least square methods.

3 Implementation of radial basis function collocation method

Consider the following form of second-order delay differential equation with discrete delay

$$\frac{d^2 y}{dx^2} = f(x, y(x), y'(x), y(px - \tau), y'(px - \tau)), \quad a \leq x \leq b, \quad (3.1)$$

subject to initial conditions $y(a) = y_0$ and $y'(a) = y'_0$, or boundary conditions $\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$ and $\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$, where $p, y_0, y'_0, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2$, and γ_2 are constants, f is a continuous linear or nonlinear function, $\tau(x, y(x))$ is the delay and $px - \tau$ is delay argument.

We can approximate the solution of Eq. (3.1) by the radial basis function networks as

$$y(x) \simeq \sum_{j=0}^M w_j h_j(x). \quad (3.2)$$

In delay equations, we also have to approximate the $y(px - \tau)$ in terms of the radial basis function series at delay time as

$$y(px - \tau) \simeq \sum_{j=0}^M w_j h_j(px - \tau). \quad (3.3)$$



Substituting Eqs. (3.2), (3.3) in Eq. (3.1), we get the residual as

$$\sum_{j=0}^M w_j \frac{d^2}{dx^2} h_j(x) - f \left(x, \sum_{j=0}^M w_j h_j(x), \sum_{j=0}^M w_j h'_j(x), \sum_{j=0}^M w_j h_j(px - \tau), \sum_{j=0}^M w_j h'_j(px - \tau) \right). \quad (3.4)$$

Set the residual (3.4) equal to zero at the set of collocation points x_i .

$$\sum_{j=0}^M w_j \frac{d^2}{dx^2} h_j(x_i) - f \left(x_i, \sum_{j=0}^M w_j h_j(x_i), \sum_{j=0}^M w_j h'_j(x_i), \sum_{j=0}^M w_j h_j(px_i - \tau), \sum_{j=0}^M w_j h'_j(px_i - \tau) \right) = 0. \quad (3.5)$$

Thus, we obtain $M - 1$ equations at collocation points x_i . It is to be noted that there are $M + 1$ unknowns, $\{w_j\}_{j=0}^M$ and $M - 1$ equations. Two more equations are needed which can be obtained from the conditions of Eq. (3.1), that is

$$y(a) = y_0, \Rightarrow \sum_{j=0}^M w_j h_j(a) = y_0,$$

$$y'(a) = y'_0, \Rightarrow \sum_{j=0}^M w_j h'_j(a) = y'_0.$$

or

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1, \Rightarrow \alpha_1 \sum_{j=0}^M w_j h_j(a) + \beta_1 \sum_{j=0}^M w_j h'_j(a) = \gamma_1,$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2, \Rightarrow \alpha_2 \sum_{j=0}^M w_j h_j(b) + \beta_2 \sum_{j=0}^M w_j h'_j(b) = \gamma_2.$$

Thus, we obtained a system of $M + 1$ equations, either linear or nonlinear, along with $M + 1$ unknown coefficients w_j . This system is solved for w_j s by Newton iterative method. One can get the approximate solution using w_j s in (3.2).

4 Implementation of proposed technique

The method of steps [5, 6] is used to transform the delay differential equations to ordinary differential equations on a given interval. Consider following class of delay differential equations with discrete delay

$$\begin{aligned} \frac{d^2 y}{dx^2} &= f(x, y(x), y'(x), y(px - \tau), y'(px - \tau)), \quad a \leq x \leq b, \\ y(x) &= \phi(x), \quad -b \leq x \leq a. \end{aligned} \quad (4.1)$$

As our method combines the method of steps and radial basis function collocation method, the proposed technique comprises two steps. In the first step, the method of steps is employed to transform the delay

differential equation to an ordinary differential equation and in the second step we implement the radial basis function collocation method, described in Sect. 3, on resulting ordinary differential equation.

Step I The solution y of the delay differential equation (4.1) is known on $[-b, a]$, say $\phi(x)$, and call this solution $y_0(x)$, that is, $y_0(px - \tau) = \phi(px - \tau)$.

Now the delay differential equation on $[a, b]$ takes the form

$$\frac{d^2 y}{dx^2} = f(x, y(x), y'(x), y_0(px - \tau), y'_0(px - \tau)), \quad a \leq x \leq b, \quad (4.2)$$

which is an ordinary differential equation with no delay term, because $y_0(px - \tau)$ and $y'_0(px - \tau)$ are known.

Step II Using the radial basis function collocation method, we solve the ordinary differential equation obtained in Step I on $[a, b]$ and denote the solution as $y_1(x)$.

Delay differential equation on $[b, 2b]$ becomes

$$\frac{d^2 y}{dx^2} = f(x, y(x), y'(x), y_1(px - \tau), y'_1(px - \tau)), \quad b \leq x \leq 2b, \quad (4.3)$$

which is again an ordinary differential equation and solve it by the radial basis function networks to get $y_2(x)$ on $[b, 2b]$. This procedure may be continued for subsequent intervals.

5 Numerical solutions

In this section, we solve linear and nonlinear delay differential equations using the radial basis function collocation method and the proposed method and compare the results with each other and exact solution. The collocation points are chosen to be the same as centres, $\{d_j\}_{j=0}^M = \{x_j\}_{j=0}^M$. The width of the j th radial basis function is considered as in Sect. 2 to be $b_j = \nu e_j$, here we consider $e_j = x_{j+1} - x_j$ and $\nu > 0$. We consider $\nu = 20, 100, 200, 200$ for Examples 1, 2, 3 and 4, respectively.

Example 1 Consider the linear delay differential equation ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}e^{\frac{x}{2}}y\left(\frac{x}{2}\right) + \frac{1}{2}y(x), \quad 0 < x \leq 1, \\ y(x) &= e^x, \quad x \leq 0. \end{aligned} \quad (5.1)$$

The exact solution is $y(x) = e^x$.

The results obtained by the radial basis function collocation method, y_{RBFN} , and the proposed method, y_{PRO} , at $M = 8$ are shown in Table 1 along with exact solution, y_{exact} , where E_{RBFN} and E_{PRO} represent the absolute error by the radial basis function collocation method and the proposed method, respectively. The proposed method provides better results as compared to the radial basis function collocation method.

Numerical results are obtained using Maple 13, a system with Core Duo CPU 2.00 GHz and 2.50 GB RAM. For the problem (5.1), proposed and radial basis function collocation methods take 7 and 9 s, respectively.

Table 1 Comparison of the radial basis function collocation method and proposed method at $M = 8$, with exact solution

x	$M = 8$				
	y_{RBFN}	y_{PRO}	y_{exact}	E_{RBFN}	E_{PRO}
0.1	1.105170918075648	1.105170863740288	1.105170855606515	6.24691E-08	5.43354E-08
0.2	1.221402758160170	1.221402856590157	1.221402863666870	1.05507E-07	9.84300E-08
0.3	1.349858807576003	1.349858788968366	1.349858782042489	2.55335E-08	1.86076E-08
0.4	1.491824697641270	1.491824438669770	1.491824412174714	2.85467E-07	2.58971E-07
0.5	1.648721270700128	1.648721317398444	1.648721323561253	5.28611E-08	4.66983E-08
0.6	1.822118800390509	1.822119254967360	1.822119295929978	4.95539E-07	4.54577E-07
0.7	2.013752707470477	2.013752509582033	2.013752499151085	2.08319E-07	1.97888E-07
0.8	2.225540928492468	2.225540178147770	2.225540113992763	8.14500E-07	7.50345E-07
0.9	2.459603111156950	2.459604456196746	2.459604485114913	1.37396E-06	1.34504E-06
1.0	2.718281828459045	2.718271894779136	2.718271375842572	1.04526E-05	9.93368E-06



Table 2 Absolute errors by radial basis function networks E_{yRBFN} , E_{zRBFN} , proposed method E_{yPRO} , E_{zPRO} , at $M = 10$

x	$M = 10$			
	E_{yRBFN}	E_{yPRO}	E_{zRBFN}	E_{zPRO}
0.1	7.0409E–10	6.7479E–10	1.3130E–10	1.4285E–10
0.2	8.3987E–10	7.3608E–10	3.1837E–11	6.2746E–11
0.3	1.0038E–09	8.0754E–10	7.0113E–11	1.7884E–11
0.4	1.2073E–09	8.9950E–10	1.7321E–10	9.6736E–11
0.5	1.4572E–09	1.0152E–09	2.8226E–10	1.7684E–10
0.6	1.7335E–09	1.1309E–09	4.0698E–10	2.6622E–10
0.7	2.0876E–09	1.2945E–09	5.3670E–10	3.5221E–10
0.8	2.4927E–09	1.4730E–09	6.8998E–10	4.5102E–10
0.9	2.9569E–09	1.6682E–09	8.6674E–10	5.5985E–10
1.0	3.3129E–09	1.7036E–09	1.1344E–09	7.4364E–10

Table 3 Absolute error by proposed method at different M

x	E_{PRO}		
	$M = 4$	$M = 8$	$M = 12$
0.1	4.83592E–08	4.81648E–12	8.87562E–19
0.2	1.75599E–06	2.32022E–11	7.60022E–19
0.3	1.75600E–05	6.06338E–12	7.35690E–18
0.4	7.95568E–05	8.69715E–12	4.81231E–20
0.5	2.49494E–04	2.09505E–10	1.97600E–17
0.6	6.27890E–04	1.35259E–10	1.32771E–17
0.7	1.36286E–03	6.19630E–10	2.59420E–18
0.8	2.65845E–03	4.02785E–09	8.29066E–17
0.9	4.78246E–03	4.63197E–08	6.46850E–16
1.0	8.07353E–03	2.28389E–07	6.32662E–14

Table 4 Absolute error by proposed method at different M

x	E_{PRO}			
	$M = 12$	$M = 15$	$M = 20$	$M = 25$
0.1	1.30716E–17	3.58196E–19	6.98392E–26	2.34956E–29
0.2	2.67614E–17	7.20286E–19	1.40681E–25	4.72857E–29
0.3	4.01346E–17	1.09121E–18	2.12345E–25	7.17179E–29
0.4	5.34373E–17	1.47850E–18	2.86677E–25	9.70581E–29
0.5	7.09090E–17	1.87855E–18	3.67347E–25	1.23424E–28
0.6	8.26400E–17	2.30394E–18	4.47129E–25	1.51001E–28
0.7	1.02193E–16	2.74542E–18	5.33450E–25	1.80139E–28
0.8	1.17716E–16	3.22784E–18	6.29137E–25	2.10847E–28
0.9	1.39402E–16	3.66054E–18	7.28024E–25	2.42230E–28

Example 2 Consider the first-order system of delay differential equations,

$$\begin{aligned} \frac{dy}{dx} &= y(x) - z(x) + y\left(\frac{x}{2}\right) - e^{\frac{x}{2}} + e^{-x}, \quad 0 \leq x \leq 1 \\ \frac{dz}{dx} &= -y(x) - z(x) - z\left(\frac{x}{2}\right) + e^{\frac{-x}{2}} + e^x, \\ y(x) &= e^x, \quad z(x) = e^{-x}, \quad 0 \leq x, \end{aligned} \quad (5.2)$$

The exact solution is [12]

$$y(x) = e^x, \quad z(x) = e^{-x}. \quad (5.3)$$

The absolute errors by the radial basis function collocation method and the proposed method are given in Table 1. According to the results obtained, the proposed method is more accurate as compared to the radial basis collocation method.

Proposed and radial basis function collocation method take 85 and 160 s, respectively, for solving System (5.2).



Example 3 Consider the delay differential equations with nonlinear delay function,

$$\begin{aligned} \frac{d^3 y}{dx^3} &= 2y^2\left(\frac{x}{2}\right) - 1, \quad 0 \leq x \leq 1, \\ y(x) &= \sin(x), \quad x \leq 0, \end{aligned} \quad (5.4)$$

The exact solution is [13], $y(x) = \sin(x)$.

Absolute error by the proposed method at different values of M is shown in Table 3. According to Table 3, we obtain more accurate results while increasing M .

Example 4 Consider the following nonlinear pantograph equation,

$$\frac{d^2 y}{dx^2} - \frac{8}{3} y' \left(\frac{x}{2} \right) y(x) - 8x^2 y \left(\frac{x}{2} \right) = \frac{-4}{3} - \frac{22}{3} x - 7x^2 - \frac{5}{3} x^3, \quad 0 \leq x \leq 1, \quad (5.5)$$

subject to the boundary conditions $y(0) = y(1) = 1$. The exact solution is given by [14], $y(x) = 1 + x - x^3$.

Table 4 shows that larger values of M give more accurate results.

6 Conclusion

Linear and nonlinear delay differential equations can easily be handled by the proposed method. It is shown that proposed method provides better results as compared to the radial basis function collocation method and are in good agreement with exact solution, as shown in Tables 1 and 2. According to Tables 3 and 4, we can get more accurate results while increasing M .

The proposed method is also time efficient than the radial basis function collocation method, as shown in Examples 1 and 2. Efficiency of radial basis function collocation method is further deteriorated in case of nonlinear delay differential equation.

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